A Survey of the Density Topology

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Outline

1. Motivation

1.1 Measure Theory

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- 2. Density
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 - 2.2 Approximate Continuity
 - 2.3 Examples

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Motivation

Suppose we have a function, continuous when restricted to a set of full measure. The continuous functions are strictly contained in this collection.

For example, consider the functions

 $\chi_{\mathbb{Q}}(x)$, the characteristic function of the rationals

 $\chi_{C}(x)$, the characteristic function of the Cantor set

$$g = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0\\ 0 & x = 0 \end{cases}$$

Lebesgue Measure

Notation: Let $\mu(A)$ denote the Lebesgue measure of a set A.

Notation: Not all sets are measurable. Let \mathcal{L} denote the collection of subsets of \mathbb{R} which are measurable.

Density

Definition

Consider a sequence of intervals centered at a point x and shrinking around that point. We define the density using the measure of these intervals. Let $d_A(x)$ denote the density of a point x with respect to a set A:

$$d_{\mathcal{A}}(x) = \lim_{n \to \infty} \frac{\mu\left(\left(x - \frac{1}{n}, x + \frac{1}{n}\right) \cap \mathcal{A}\right)}{\mu\left(\left(x - \frac{1}{n}, x + \frac{1}{n}\right)\right)}$$

provided the limit exists.

Let A = (0,1). Every x ∈ A has density d_A(x) = 1.
 *This holds for any open set A.

2. Consider $y \in A^c$. If $y \neq 0, 1$, then $d_A(y) = 0$. If y = 0 or 1, then $d_A(y) = \frac{1}{2}$.

3. Let $B = \{1, \frac{1}{2}, \frac{1}{3}, ...\} \cup \{0\}$. Then for each $x \in B$, we have $d_B(x) = 0$. Actually, $d_{B^c}(x) = 1$.

Remark: Note that if a point has positive density in a set A, then A has positive measure.

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Density Function

Definition

The function $\Phi(A) : \mathcal{L}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ is called the *density function* on \mathbb{R} because it sends each set $A \in \mathcal{L}$ to the set of points with density 1 in A.

$$\Phi(A) = \{x \in \mathbb{R} : d_A(x) = 1\}$$

*There are extensions of Φ such that $\Phi(A)$ is defined for some non-measurable A. Such an extension is also called a density function, but is outside the scope of this presentation.

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Lebesgue Density Theorem

Theorem

Let $E \in \mathcal{L}$. Then $\mu(E\Delta\Phi(E)) = 0$. That is, for any measurable set, the symmetric difference of E and $\Phi(E)$ is a nullset.

Outline of Proof (Faure).

Consider the sets $A_n = \{x \in E \cap (-n, n) : d_E(x) < \frac{n}{n+1}\}$. Then as $n \to \infty$, the sets will expand to cover all of \mathbb{R} , and the density will approach 1. Notice that $E \setminus \Phi(E) = \bigcup_n A_n$; if we can show $\mu(A_n) = 0$, then $\mu(E \setminus \Phi(E)) = 0$.

Cover A_n with an open set U such that $\mu(U) < \mu(A_n) + \varepsilon$. It takes some proof to show that we can choose U such that $\mu(U) < n\varepsilon$ for any given ε . Since ε is arbitrary, A_n is a nullset. The countable union $\bigcup_{n=1}^{\infty} A_n$ is a nullset.

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Approximate Continuity

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is approximately continuous at x_0 iff there exists a set $A \in \mathcal{L}$ such that $x_0 \in \Phi(A)$ and

$$\lim_{x\to x_0, x\in A} f(x) = f(x_0)$$

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1. Define $f(x) = \begin{cases} 1 & x = 1, \frac{1}{2}, \frac{1}{3}, \dots \\ 0 & \text{otherwise} \end{cases}$. Then f(x) is approximately continuous for all $x \neq 1, \frac{1}{2}, \frac{1}{3}, \dots$.

 Consider χ_Q, the characteristic function of the rationals. χ_Q(x) = 1 if x ∈ Q, 0 otherwise. χ_Q is nowhere continuous, but it is approximately continuous for each x ∉ Q.

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$$g(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0\\ 0 & x = 0 \end{cases}$$

4. The collection of measurable functions is the same as the collection of approximately continuous functions.

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Density Topology

Definition

We define the *density topology* on \mathbb{R} as follows: A set A is open in the density topology iff A is measurable and $A \subseteq \Phi(A)$, where $\Phi(A)$ is the set of density points of A. We denote the topology $(\mathbb{R}, \mathcal{T})$.

Remark: The union of measurable sets need not be measurable. It requires some proof to show that $A_i \in \mathcal{L}$ and A_i measurable $\Rightarrow \bigcup A_i \in \mathcal{L}$ and $\bigcup A_i \subseteq \Phi(\bigcup A_i)$.

Remark: A function $f : \mathbb{R} \to \mathbb{R}$ is approximately continuous iff f takes open sets in \mathcal{T} to open sets in the usual topology.

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| Set | $\Phi(Set)$ | Euclidean | Density Topology |
|---|-------------|-----------|------------------|
| (0,1) | (0,1) | open | open |
| $(0,1)\cup(1,2)$ | (0,2) | open | open |
| $(-1,1)\setminus \{\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\}$ | (-1,1) | not open | open |
| $(0,1)\setminus \mathbb{Q}$ | (0,1) | not open | open |
| $(0,1)\cup\{2\}$ | (0,1) | not open | not open |

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| $(0,1)\cup\{2\}$ | (0,1) | not open | not open |

Remark: The density topology is finer than the Euclidean topology on \mathbb{R} .

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Topological Properties

(\mathbb{R},\mathcal{T}) is a Hausdorff (\mathcal{T}_2) space: The density topology is finer than the usual topology.

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$(\mathbb{R}, \mathcal{T})$ is not Separable

Theorem

The space $(\mathbb{R}, \mathcal{T})$ is not separable.

Proof by Contradiction.

Assume that $(\mathbb{R}, \mathcal{T})$ is separable. Then there exists a countable dense subset $V \subseteq \mathbb{R}$. Consider the set $U = \mathbb{R} \setminus V$. Since $\mu(V) = 0$, we have $U \subseteq \Phi(U)$. So U is open and V is not dense.

(\mathbb{R},\mathcal{T}) is not First Countable

Definition

A space is *first countable* if for each point x there exists a countable sequence of open neighborhoods of x: N_1, N_2, \ldots such that given an open neighborhood U of x we can find an $N_i \subseteq U$ for some *i*.

Proof by Contradiction.

Assume that $(\mathbb{R}, \mathcal{T})$ is first countable. Fix $x \in \mathbb{R}$. Choose some $y_i \in N_i$ such that $y_i \neq x$ for each $i < \infty$. Then let $U = N_1 \setminus \{y_i : i < \infty\}$. Since $\mu(\{y_i\}) = 0$, we know that $U \subseteq \Phi(U)$ so U is open in \mathcal{T} . Then notice that $N_i \not\subseteq U$ for each i.

(\mathbb{R},\mathcal{T}) is not First Countable

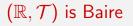
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Lemma

A set A is a nullset iff A is nowhere dense in \mathcal{T} .

Proof.

Let A be a nullset. Then $\Phi(A^c) = \mathbb{R}$, so $\Phi(A) = \emptyset$. But $A^c \subseteq \Phi(A^c) = \mathbb{R}$, so A^c is an open set. Then A is closed. Also, $Int(A) \subseteq \Phi(A) \cap A = \emptyset$. So A is a closed set with empty interior, and A is nowhere dense.

Let A be nowhere dense. Then $\Phi(A) = \emptyset$. Using the Lebesgue Density Theorem: $\mu(A\Delta\Phi(A)) = 0$ gives us $\mu(A) = 0$.



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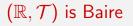
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Definition

A topological space is a *Baire space* if every union of countably many closed nowhere dense sets has empty interior.

Proof.

Nowhere dense sets are nullsets. Countable unions of nullsets are nullsets. Nullsets are nowhere dense. So the countable union has empty interior.

(\mathbb{R},\mathcal{T}) is Baire

Definition

A topological space is a *Baire space* if every union of countably many closed nowhere dense sets has empty interior.

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Further Study

How can we translate the density topology into other spaces?

What are the necessary and sufficient conditions for a measure space to support the density topology?

What does the density topology look like on the space C[0, 1]?

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